Geometry of coupled dispersionless equations by using bishop frames

Kemal Eren

Department of Mathematics, Faculty of Arts and Sciences,Sakarya University, Sakarya, Turkey
E-mail: kemaleren52@gmail.com

Abstract

In this study, we investigate the equations of the coupled dispersionless and complex coupled dispersionless by an alternative point of view to avoid insufficiency occurs for sudden orientation changes when the curve straightens out momentarily by considering a moving coordinate frame on a space curve based on the concept of parallel transport. Firstly, we study the correlation between the type-1 Bishop frame of a space curve and the coupled dispersionless equation: then we discover the correlation between the type-2 Bishop frame and the complex coupled dispersionless equations. Finally, we give the integrability conditions of the obtained differential equations by finding the Lax pair.

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1 Introduction

Most of the integrable systems arise as a quasi-classical limit of ordinary integrable systems with a dispersion term. There are however important examples of integrable systems which are referred to as dispersionless, not in this sense, but because the dispersion term is absent. The class of integrable partial differential equations called dispersionless integrable systems have attracted much interest in view of their wide range of applications in various fields of mathematics and physics.

The development process of the dispersionless systems can be seen in the table below;

Table 1. Dispersionless Equations

Type of Dispersionless Equations	**Abbreviations**	**Equation systems**	**References**
Coupled Dispersionless	CD	$\rho_s + uu_y = 0,$ $u_{ys} = \rho u,$	[1, 2]
Generalized Coupled Dispersionless	GCD	$\rho_s + \frac{(uv)_y}{2} = 0,$ $u_{ys} = \rho u,$ $v_{ys} = \rho v,$	[3]
Complex Couple Dispersionless	CCD	$\rho_s + \frac{1}{2}\left(\lvert u\rvert^2\right)_y = 0,$ $u_{ys} = \rho u.$	[4]

These equation systems have been generalized by many researchers in many different aspects and proposed as models for describing physical phenomena [5]-[16]. For instance, Shen, et al. have studied the CCD equation and the complex short pulse equation from the algebraic and geometric point of view and they have exposed the link of the motions of space curves to the real and complex

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CD equations and short pulse equations via hodograph transformations by using the Serret-Frenet frame of the curves in [17].

On the other hand, the theory of curves is primarily formed by the Serret-Frenet formulas which describe the kinematic properties of a particle moving along a differentiable curve or the geometric properties of the curve itself irrespective of any motion. A great deal of research has been conducted on the Frenet frame formalism of space curves. However, it is known that it is not possible to calculate the Frenet frame for straight-line segments and at points where the second derivative vanishes. So, there is an alternative way to define the moving reference frame that was introduced by L.R. Bishop in 1975 by means of parallel vector fields and so-called as alternative or parallel frame of the curves [18]. Recently, some researchers called this frame as Bishop (parallel transport) frame and this frame based studies have been intensively treated, see [19, 20, 21, 22]. In these regards, we study the link of the coupled dispersionless equations and short pulse equations with space curves according to Bishop frames.

2 Preliminaries

Let $\gamma=\gamma(s)$ be a regular unit speed curve in Euclidean 3-space. If T, N, and B denote the tangent, principal normal and binormal unit vectors at $\gamma(s)$ point of the curve γ, respectively. Then the Frenet formulas are given

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

where $\kappa=\|T'\|$ is curvature and $\tau=-\langle N,B'\rangle$ is torsion of the curve γ. This classical frame $\{T,N,B\}$ provides to illustrate a lot of properties of a curve; however, it is not defined for all points along every curve. The alternative way of defining a moving reference frame is Bishop frame. The Bishop frame provides significant advantages for curves whose second derivative is zero at some points according to the Frenet frame. This frame is constructed in two different ways as the type-1 Bishop frame $\{T,N_1,N_2\}$ and the type-2 Bishop frame $\{M_1,M_2,B\}$. Let's give the relations between the Frenet frame and the Bishop frames.

The type-1 Bishop frame $\{T,N_1,N_2\}$ is based on the model where T is the tangent vector and any convenient arbitrary basis is $\{N_1,N_2\}$ for the normal plane perpendicular to the tangent vector. Then the derivative equations of type-1 Bishop frame are given by

$$\begin{bmatrix} T' \\ {N_1}' \\ {N_2}' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix} \tag{2.1}$$

where $\langle T,T\rangle=\langle N_1,N_1\rangle=\langle N_2,N_2\rangle=1$, $\langle T,N_1\rangle=\langle T,N_2\rangle=\langle N_1,N_2\rangle=0$. Also k_1 and k_2 are called the curvatures of the curve according to the type-1 Bishop frame. Let θ be the angle between the vectors N_1 and N of γ, then the relation between the Frenet frame and the type-1 Bishop frame is

$$\begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

Moreover, the relations between the curvatures of curve according to Frenet and type-1 Bishop frames are given

$$\kappa = \sqrt{k_1^2 + k_2^2},\ \tau = \frac{d\theta}{ds}$$

or

$$k_1 = \kappa \cos\theta, k_2 = \kappa \sin\theta, \theta = \arctan\left(\frac{k_2}{k_1}\right) = \int \tau ds.$$

On the other hand, let us denote type-2 Bishop frame of a curve γ by $\{M_1, M_2, B\}$ then the derivative equations of type-2 Bishop frame are given as

$$\begin{bmatrix} M_1{}' \\ M_2{}' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\varepsilon_1 \\ 0 & 0 & -\varepsilon_2 \\ \varepsilon_1 & \varepsilon_2 & 0 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ B \end{bmatrix} \tag{2.2}$$

such that $\langle M_1, M_1\rangle = \langle M_2, M_2\rangle = \langle B, B\rangle = 1$, $\langle M_1, M_2\rangle = \langle M_1, B\rangle = \langle M_2, B\rangle = 0$. Here ε_1 and ε_2 denote the curvatures of the curve according to the type-2 Bishop frame. Let φ be the angle between the vectors N_2 and N, then there is the relation between the Frenet frame and the type-2 Bishop frame as

$$\begin{bmatrix} M_1 \\ M_2 \\ B \end{bmatrix} = \begin{bmatrix} \sin\varphi & \cos\varphi & 0 \\ -\cos\varphi & \sin\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

The relation between the curvatures of curve according to the Frenet frame and Bishop-2 frame is

$$\tau = \sqrt{\varepsilon_1^2 + \varepsilon_2^2},\ \ \kappa = \frac{d\varphi}{ds}$$

or

$$\varepsilon_1 = -\tau\cos\varphi, \varepsilon_2 = -\tau\sin\varphi, \varphi = \arctan\left(\frac{\varepsilon_2}{\varepsilon_1}\right) = \int \kappa ds.$$

3 The link of the coupled dispersionless equation with space curves according to Bishop frames

In this part of the study, we investigate the correlation between the Bishop frames of space curves and the equations of CD. Let's assume that

$$\gamma(y, s) : [0, l]\, x\, [0, S] \to E^3$$

is a family of space curves, where $y \in [0, l]$ is the arc-length parameter and s represents the time.

Theorem 3.1. Let $\gamma(y, s)$ be a family of space curves, then the following statement provides the CD equation

$$\begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}_s = \begin{bmatrix} 0 & -c^{-1} & 0 \\ c^{-1} & 0 & -u \\ 0 & u & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}, \tag{3.1}$$

where $\{T, N_1, N_2\}$ is the type-1 Bishop frame, c is a non-zero constant, u is a real function.

Proof. We obtain time evolution for the orthogonal frame $\{T, N_1, N_2\}$ in matrix form as

$$\begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}_s = \begin{bmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \delta \\ -\beta & -\delta & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix} \tag{3.2}$$

where α, β and δ are functions of y and s. Also, from the type-1 Bishop formulae given by (2.1), we write

$$\begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}_y = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}.$$

Thus, under favorable conditions $T_{sy} = T_{ys}, N_{1,sy} = N_{1,ys}, N_{2,sy} = N_{2,ys}$ we have

$$\alpha_y = k_{1,s} - k_2\delta, \tag{3.3}$$

$$\beta_y = k_1\delta + k_{2,s}, \tag{3.4}$$

$$\delta_y = -k_1\beta + k_2\alpha. \tag{3.5}$$

By the hypothesis $\alpha = -c^{-1}$, $\beta = 0$ and $\delta = -u$ are satisfied and then the equations (3.3)-(3.5) become

$$0 = k_{1,s} + k_2 u, \tag{3.6}$$

$$0 = -k_1 u + k_{2,s}, \tag{3.7}$$

$$u_y = k_2 c^{-1}, \tag{3.8}$$

respectively. So, from the equation (3.8) we easily see that

$$k_2 = c u_y. \tag{3.9}$$

By substituting the equation (3.9) into the equation (3.6), we find

$$k_1 = c\rho \tag{3.10}$$

under consideration $\rho_s = -u u_y$. By substituting the equations (3.9) and (3.10) into the equations (3.6) and (3.7), respectively, we have

$$\rho_s + u u_y = 0 \tag{3.11}$$

and

$$u_{ys} = \rho u. \tag{3.12}$$

As a result, the equations (3.11) and (3.12) express the CD equation and this completes the proof.

Q.E.D.

Corollary 3.2. The CD equation corresponds to the set

$$\{k_1, k_2, \alpha, \beta, \delta\} = \{c\rho, cu_y, -c^{-1}, 0, -u\}.$$

In 1968 P.D. Lax introduced the Lax equations providing the integrability of nonlinear differential equations [2, 23]. Now, let's give the Lax pair which provides integrability of CD equation by the following theorem.

Theorem 3.3. Let's take a function $\psi = \psi(y,s)$ with $\mathrm{so}(3)$ value, then Lax pair of the CD equation is

$$\psi_y = U\psi, \psi_s = V\psi \tag{3.13}$$

such that $U = -k_1e_3 + k_2e_2$ and $V = -\alpha e_3 - \delta e_1$, where $k_1 = \langle T_y, N_1\rangle$, $k_2 = \langle T_y, N_2\rangle$, $\alpha = \langle T_s, N_1\rangle$, $\delta = \langle N_{1s}, N_2\rangle$ and $\{T, N_1, N_2\}$ is the type-1 Bishop frame of the space curve $\gamma(y,s)$.

Proof. The Lax pair of the CD equation is

$$U = -i\lambda \begin{pmatrix} \rho & u_y \\ u_y & -\rho \end{pmatrix}, V = \begin{pmatrix} \frac{i}{4\lambda} & \frac{-u}{2} \\ \frac{u}{2} & -\frac{i}{4\lambda} \end{pmatrix}. \tag{3.14}$$

Also, the compatibility condition $U_y - V_s + UV - VU = 0$ satisfies the CD equation [3]. The basis of $\mathrm{su}(2)$ and $\mathrm{so}(3)$ are

$$e_1 = \frac{1}{2i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathrm{e}_2 = \frac{1}{2i}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \mathrm{e}_3 = \frac{1}{2i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.15}$$

and

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.16}$$

respectively [24] and there is an isomorphism $L_j \to e_j$, $(j = 1, 2, 3)$ between the Lie algebras $\mathrm{su}(2)$ and $\mathrm{so}(3)$. Under this isomorphism, the CD equation provides the equation (3.13). From the corollary 3.2 and hypothesis, the functions U and V are found as

$$\begin{aligned} U &= -k_1e_3 - k_2e_1 = -c\rho e_3 - cu_ye_1 = -i\lambda \begin{pmatrix} \rho & u_y \\ u_y & -\rho \end{pmatrix}, \\ V &= -\alpha e_3 - \delta e_2 = c^{-1}e_3 + ue_2 = \begin{pmatrix} \frac{i}{2\lambda} & \frac{-u}{2} \\ \frac{u}{2} & \frac{-i}{2\lambda} \end{pmatrix}, \end{aligned}$$

where $c = -2\lambda$. They provide a Lax pair of the CD equation as it is desired. Q.E.D.

Let us give the geometric interpretation of the conserved quantity of the CD equation by the following theorem.

Theorem 3.4. The conserved quantity of the CD equation is constant,

$$I = \rho^2 + u_y^2$$

where $\rho = \frac{k_1}{c}$, $u_y = \frac{k_2}{c}$, k_1 and k_2 are the type-1 Bishop curvatures of the space curve $\gamma(y, s)$.

Proof. Considering the equations (3.6) and (3.7), we find

$$\frac{d}{ds}\left({k_1}^2 + {k_2}^2\right) = 2k_1 k_{1,s} + 2k_2 k_{2,s} = 2k_1\left(-uk_2\right) + 2k_2\left(uk_1\right) = 0.$$

On the other hand, from the equation (3.9) and (3.10), we get

$$\frac{d}{ds}\left({k_1}^2 + {k_2}^2\right) = \frac{d}{ds}\left(c^2\rho^2 + (cu_y)^2\right) = c^2\frac{d}{ds}\left(\rho^2 + u_y^2\right).$$

As a result, for $c \neq 0$, we find out

$$\frac{d}{ds}\left(\rho^2 + u_y^2\right) = 0.$$

Hence, we can easily see that the conserved quantity of the CD equation is constant. Q.E.D.

Corollary 3.5. If u is a real function of y and s then the CD equation is obtained by condition (3.1).

Theorem 3.6. Let $\gamma(y, s)$ be a space curve family, then the following statements provide the CCD equation.

$$\begin{bmatrix} M_1 \\ M_2 \\ B \end{bmatrix}_s = \begin{bmatrix} 0 & -c^{-1} & u_1 \\ c^{-1} & 0 & -u_2 \\ -u_1 & u_2 & 0 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ B \end{bmatrix} \tag{3.17}$$

where $\{M_1, M_2, B\}$ is the type-2 Bishop frame, $c \neq 0$ is constant, $u = u_1 + iu_2$ is a complex function.

Proof. We obtained time evolution for the orthogonal frame $\{M_1, M_2, B\}$ in matrix form as

$$\begin{bmatrix} M_1 \\ M_2 \\ B \end{bmatrix}_s = \begin{bmatrix} 0 & \eta & \mu \\ -\eta & 0 & \omega \\ -\eta & -\omega & 0 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ B \end{bmatrix} \tag{3.18}$$

where η, μ and ω are functions of y and s. Also from the type-2 Bishop formulae given by (2.2), we write

$$\begin{bmatrix} M_1 \\ M_2 \\ B \end{bmatrix}_y = \begin{bmatrix} 0 & 0 & -\varepsilon_1 \\ 0 & 0 & -\varepsilon_2 \\ \varepsilon_1 & \varepsilon_2 & 0 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ B \end{bmatrix}.$$

Thus, under favorable conditions $T_{sy} = T_{ys}, M_{1,sy} = M_{1,ys}, M_{2,sy} = M_{2,ys}$ we have

$$\eta_y = \varepsilon_1\omega - \varepsilon_2\mu, \tag{3.19}$$

$$\mu_y = \varepsilon_2\eta - \varepsilon_{1,s}, \tag{3.20}$$

$$\omega_y = -\varepsilon_{2,s} - \varepsilon_1 \eta. \tag{3.21}$$

From the equations (3.20) and (3.21), we obtain

$$(\varepsilon_1 - i\varepsilon_2)_s = i\eta (\varepsilon_1 - i\varepsilon_2) - (\mu - i\omega)_y. \tag{3.22}$$

Let us assume that $\varepsilon_1 - i\varepsilon_2 = icu_y, \eta = -c^{-1}, \mu - i\omega = u$ in this equation, then we have

$$\varepsilon_1 = -cu_{2y}, \varepsilon_2 = -cu_{1y} \tag{3.23}$$

and

$$\eta = -c^{-1}, \mu = u_1, \omega = -u_2. \tag{3.24}$$

By substitution the equations (3.23) and (3.24) into the equations (3.22), we have

$$u_{ys} = 0 \tag{3.25}$$

under consideration $\rho = 0$. By substituting the equations (3.23) and (3.24) into the equations (3.19), we have

$$u_1 u_{1y} + u_2 u_{2y} = 0. \tag{3.26}$$

On the other hand, we find

$$\rho_s + \frac{1}{2}\left(|u|^2\right)_y = \frac{1}{2}\left(uu_y^* + u_y u^*\right) = u_1 u_{1y} + u_2 u_{2y}. \tag{3.27}$$

From the equations (3.26) and (3.27), we obtain

$$\rho_s + \frac{1}{2}\left(|u|^2\right)_y = 0. \tag{3.28}$$

As a result, from (3.25) and (3.28) we obtained the CCD equation Thus, the theorem is proved.

Q.E.D.

Corollary 3.7. The CCD equation corresponds to the set

$$\{\varepsilon_1, \varepsilon_2, \eta, \mu, \omega\} = \left\{-cu_{2y}, -cu_{1y}, -c^{-1}, u_1, u_2\right\}.$$

Let's give the Lax pair which provides integrability of CCD equation by the theorem.

Theorem 3.8. Let's take the function $\psi(y, s)$ with $so(3)$ value, then Lax pair of the CCD equation is

$$\psi_y = P\psi, \psi_s = Q\psi \tag{3.29}$$

such that $P = -\varepsilon_1 e_1 + \varepsilon_2 e_1$ and $Q = -\eta e_3 + \mu e_2 - \omega e_1$, where $\varepsilon_1 = -\langle M_1, B\rangle$, $\varepsilon_2 = -\langle M_2, B\rangle$, $\eta = \langle M_1, M_2\rangle$, $\mu = \langle M_1, B\rangle$, $\omega = \langle M_2, B\rangle$ and $\{M_1, M_2, B\}$ is the type-2 Bishop frame of the space curve $\gamma(y, s)$.

Proof. The Lax pair of the CCD equation is given by [4] as follows

$$P=-i\lambda\begin{pmatrix}0 & u_y\\ u_y^* & 0\end{pmatrix},\mathrm{Q}=\begin{pmatrix}\frac{i}{4\lambda} & \frac{-u}{2}\\ \frac{u^*}{2} & -\frac{i}{4\lambda}\end{pmatrix}. \tag{3.30}$$

Also, the compatibility condition $P_y-Q_s+PQ-QP=0$ satisfies the CCD equation and u^* is the complex conjugate of u. From the equations (3.15) and (3.16), the basis of $\mathrm{su}\,(2)$ and $\mathrm{so}\,(3)$ are $\{e_1, \mathrm{e}_2, \mathrm{e}_3\}$ and $\{L_1, L_2, L_3\}$, respectively. Let $L_j\to e_j$, $(j=1,2,3)$ be an isomorphism between the Lie algebras $\mathrm{su}\,(2)$ and $\mathrm{so}\,(3)$, Under this isomorphism, the CCD equation provides the equation (3.29). According to type-2 Bishop frame, P and Q functions are found as

$$\begin{array}{l} P=-\varepsilon_1 e_1+\varepsilon_2 e_1=\frac{1}{2}\begin{pmatrix}0 & \varepsilon_1-i\varepsilon_2\\ -\varepsilon_1-i\varepsilon_2 & 0\end{pmatrix}=-i\lambda\begin{pmatrix}0 & u_y\\ u_y^* & 0\end{pmatrix}\\ Q=-\eta e_3+\mu e_2-\omega e_1=\frac{1}{2}\begin{pmatrix}i\eta & -\mu+i\omega\\ \mu+i\omega & -i\eta\end{pmatrix}=\begin{pmatrix}\frac{i}{4\lambda} & \frac{-u}{2}\\ \frac{u^*}{2} & \frac{-i}{4\lambda}\end{pmatrix}\end{array}, \tag{3.31}$$

where $c=-2\lambda$, P and Q provide $P_y-Q_s+PQ-QP=0$. Q.E.D.

Let us give the geometric interpretation of the conserved quantity of the CCD equation with the following theorem:

Theorem 3.9. The conserved quantity of the CCD equation is constant,

$$I=\rho^2+|u_y|^2$$

where $u_{2y}=-\frac{\varepsilon_1}{c}$, $u_{1y}=-\frac{\varepsilon_2}{c}$, ε_1 and ε_2 are the type-2 Bishop curvature of the space curve family $\gamma(y,s)$.

Proof. Considering the equations (3.20), (3.21), (3.23) and (3.24), we find

$$\frac{d}{ds}\left({k_1}^2+{k_2}^2\right)=2k_1k_{1,s}+2k_2k_{2,s}=2cu_{1y}u_{2y}-2cu_{1y}u_{2y}=0.$$

On the other hand, from the equation (3.23), we get

$$\frac{d}{ds}\left({k_1}^2+{k_2}^2\right)=\frac{d}{ds}\left((-cu_{2y})^2+(-cu_{1y})^2\right)=c^2\frac{d}{ds}\left((u_{2y})^2+(u_{1y})^2\right)=c^2\frac{d}{ds}\left(|u|^2\right)_y.$$

As a result, when the last two equations are equalized for $c\neq 0$, we get

$$\frac{d}{ds}\left(|u|^2\right)_y=\frac{d}{ds}\left(\rho^2+|u|^2\right)_y=0.$$

Hence, we can easily see that the conserved quantity of the CCD equation is constant. Q.E.D.

Corollary 3.10. If $u=u_1+iu_2$ is a complex function of y and s then the CCD equation obtained by condition (3.17).

Corollary 3.11. By means of the hodograph (reciprocal) transformation $((y,s)\to(x,t))$

$$x=y_0+\int\limits_{y_0}^{y}\rho\,(y',s')dy',\quad t=s,$$

the equation CSP is obtained from CCD equation according to Bishop frame [25].

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